Introduction to regular perturbation theory

Very often, a mathematical problem cannot be solved exactly or, if the exact solution is available, it exhibits such an intricate dependency in the parameters that it is hard to use as such. It may be the case, however, that a parameter can be identified, say ε, such that the solution is available and reasonably simple for $\varepsilon = 0$. Then, one may wonder how this solution is altered for non-zero but small ε. Perturbation theory gives a systematic answer to this question.

Perturbation theory for algebraic equations. Consider the quadratic equation

$$x^2 - 1 = \varepsilon x. \tag{1}$$

The two roots of this equation are

$$x_1 = \varepsilon/2 + \sqrt{1 + \varepsilon^2/4}, \qquad x_2 = \varepsilon/2 - \sqrt{1 + \varepsilon^2/4}. \tag{2}$$

For small ε, these roots are well approximated by the first few terms of their Taylor series expansion (see figure 1)[1]

$$x_1 = 1 + \varepsilon/2 + \varepsilon^2/8 + O(\varepsilon^3), \qquad x_2 = -1 + \varepsilon/2 - \varepsilon^2/8 + O(\varepsilon^3). \tag{3}$$

Can we obtain (3) without prior knowledge of the exact solutions of (1)? Yes, using regular perturbation theory. The technique involves four steps.

STEP A. Assume that the solution(s) of (1) can be Taylor expanded in ε. Then we have

$$x = X_0 + \varepsilon X_1 + \varepsilon^2 X_2 + O(\varepsilon^3), \tag{4}$$

for X_0, X_1, X_2 to be determined.

STEP B. Substitute (4) into (1) written as $x^2 - 1 - \varepsilon x = 0$, and expand the left hand side of the resulting equation in power series of ε. Using

$$\begin{aligned} x^2 &= X_0^2 + 2\varepsilon X_0 X_1 + \varepsilon^2 (X_1^2 + 2X_0 X_2) + O(\varepsilon^3), \\ \varepsilon x &= \varepsilon X_0 + \varepsilon^2 X_1 + O(\varepsilon^3), \end{aligned} \tag{5}$$

[1] $a(\epsilon) = O(b(\epsilon))$ as $\varepsilon \to 0$, ("$a(\varepsilon)$ is big-oh of $b(\epsilon)$") if there exists a positive constant M such that $|a(\varepsilon)| \leq M|b(\varepsilon)|$ whenever ε is sufficiently close to 0.

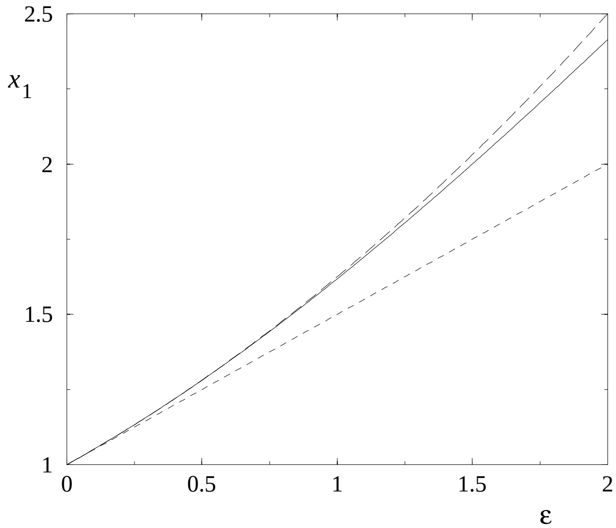

Figure 1: The root x_1 plotted as a function of ε (solid line), compared with the approximations by truncation of the Taylor series at $O(\varepsilon^2)$, $x_1 = 1 + \varepsilon/2$ (dotted line), and $O(\varepsilon^3)$, $x_1 = 1 + \varepsilon/2 + \varepsilon^2/8$ (dashed line). Notice that even though the approximations are *a priori* valid in the range $\varepsilon \ll 1$ only, the approximation $x_1 = 1 + \varepsilon/2 + x^2/8$ is fairly good even up to $\varepsilon = 2$.

this gives

$$X_0^2 - 1 + \varepsilon(2X_0X_1 - X_0) + \varepsilon^2(X_1^2 + 2X_0X_2 - X_1) + O(\varepsilon^3) = 0. \tag{6}$$

STEP C. Equate to zero the successive terms of the series in the left hand side of (6):

$$\begin{aligned} O(\varepsilon^0): &\qquad X_0^2 - 1 = 0,\\ O(\varepsilon^1): &\qquad 2X_0X_1 - X_0 = 0,\\ O(\varepsilon^2): &\qquad X_1^2 + 2X_0X_2 - X_1 = 0,\\ O(\varepsilon^3): &\qquad \cdots \end{aligned} \tag{7}$$

STEP D. Successively solve the sequence of equations obtained in (7). Since $X_0^2 - 1 = 0$ has two roots, $X_0 = \pm 1$, one obtains

$$\begin{aligned} X_0 &= 1, & X_1 &= 1/2, & X_2 &= 1/8,\\ X_0 &= -1, & X_1 &= 1/2, & X_2 &= -1/8. \end{aligned} \tag{8}$$

It can be checked that substituting (8) into (4) one recovers (3).

From the previous example it might not be clear what the advantage of regular perturbation theory is, since one can obtain (3) more directly by Taylor expansion of the roots in (2). To see the strength of regular perturbation theory, consider the following equation

$$x^2 - 1 = \varepsilon e^x. \tag{9}$$

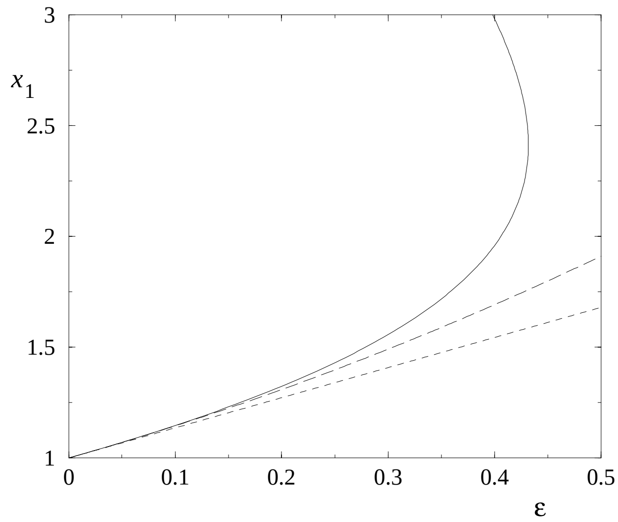

Figure 2: The solid line is the graph of two (why two?) of the three solutions of (9) obtained numerically and plotted as a function of ε (solid line). Also plotted are the approximations by truncation of the Taylor series at $O(\varepsilon^2)$, $x_1 = 1 + \varepsilon e/2$ (dotted line), and $O(\varepsilon^3)$, $x_1 = 1 + \varepsilon e/2 + \varepsilon^2 e^2/8$ (dashed line) (see (14)).

The solutions of this equation are not available; therefore the direct method is inapplicable here. However, the Taylor series expansion of these solutions can be obtained by perturbation theory. We introduce the expansion (4) as in Step A. In Step B, we use (recall that $e^z = 1 + z + z^2/2 + O(z^3)$)

$$\varepsilon e^x = \varepsilon e^{X_0+\varepsilon X_1+\varepsilon^2 X_2+O(\varepsilon^3)} = \varepsilon e^{X_0} e^{\varepsilon X_1+\varepsilon^2 X_2+O(\varepsilon^3)} = \varepsilon e^{X_0} + \varepsilon^2 X_1 e^{X_0} + O(\varepsilon^3). \tag{10}$$

Substituting this expression in (9) written as $x^2 - 1 - \varepsilon e^x = 0$ and using (5), we obtain

$$X_0^2 - 1 + \varepsilon\left(2X_0X_1 - e^{X_0}\right) + \varepsilon^2\left(X_1^2 + 2X_0X_2 - X_1e^{X_0}\right) + O(\varepsilon^3) = 0. \tag{11}$$

Thus, the sequence of equations obtained in Step C is

$$\begin{aligned}
O(\varepsilon^0): &\qquad X_0^2 - 1 = 0,\\
O(\varepsilon^1): &\qquad 2X_0X_1 - e^{X_0} = 0,\\
O(\varepsilon^2): &\qquad X_1^2 + 2X_0X_2 - X_1e^{X_0} = 0,\\
O(\varepsilon^3): &\qquad \cdots
\end{aligned} \tag{12}$$

from which we obtain (step D)

$$\begin{aligned}
X_0 &= 1, &\qquad X_1 &= e/2, &\qquad X_2 &= e^2/8,\\
X_0 &= -1, &\qquad X_1 &= -1/(2e), &\qquad X_2 &= -1/(8e^2),
\end{aligned} \tag{13}$$

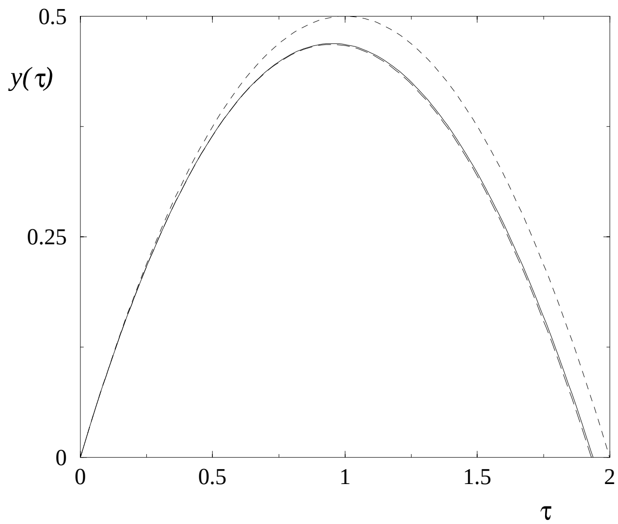

Figure 3: The exact solution (17) (solid) line is compared with the approximations by truncation of the Taylor series (see (18)) at $O(\varepsilon)$ (dotted line), $O(\varepsilon^2)$ (dashed line), and $O(\varepsilon^3)$ (indistinguishable from solid line).

or, equivalently,

$$\begin{aligned} x_1 &= 1 + \varepsilon e/2 + \varepsilon^2 e^2/8 + O(\varepsilon^3) \\ x_2 &= -1 - \varepsilon/(2e) - \varepsilon^2/(8e^2) + O(\varepsilon^3). \end{aligned} \tag{14}$$

The expression for x_1 is compared to the numerical solution of (9) on figure 2.

Remark: In fact (9) has three solutions for $0 < \varepsilon < \varepsilon_1$, with $\varepsilon_1 \approx 0.43$, and only one for $\varepsilon > \varepsilon_1$. The solution which exists for all $\varepsilon > 0$ is the one with expansion given in x_2 in (14); the solution with the expansion given in x_1 in (14) disappears for $\varepsilon > \varepsilon_1$; and the third solution (see figure 2: the solid line is the graph of a two-valued function) cannot be obtained by regular perturbation.

Exercise 1. Solve by perturbation

$$x^2 - 4 = \varepsilon \ln(x). \tag{15}$$

Notice that, as $\varepsilon \to 0$, (15) formally reduces to the equation $x^2 - 4 = 0$, with two roots $x_{1,2} = \pm 2$. (15) also has two solutions (why?). How are they related to $x_{1,2} = \pm 2$? Can both solutions of (15) be obtained by perturbation?

Perturbation theory for differential equations. Consider

$$\frac{d^2y}{d\tau^2} = -\varepsilon\frac{dy}{d\tau} - 1, \qquad y(0) = 0, \qquad \frac{dy}{d\tau}(0) = 1. \tag{16}$$

Recall that this equation governs the dynamics of a projectile thrown vertically into the air if air friction is taken into account. Here $\varepsilon = kV/(mg)$, where V is the initial velocity of the projectile, m is its mass, k is the friction constant ($[k] = MT^{-1}$) and g is the acceleration from gravity. In (16), the altitude is measured in units of $\ell_1 = V^2/g$, and the time in units of $\tau_1 = V/g$.

The exact solution of (16) is (can you show this?)

$$y(\tau) = \frac{(1+\varepsilon)}{\varepsilon^2}\left(1 - e^{-\varepsilon\tau}\right) - \frac{\tau}{\varepsilon}. \tag{17}$$

For $0 < \varepsilon \ll 1$, which corresponds to a situation where air friction is small, we can approximate $y(\tau)$ by the first few terms of its Taylor series expansion in ε (see figure 3). Using $1 - e^{-z} = z - z^2/2 + z^3/3! + O(z^4)$, we obtain

$$y(\tau) = \tau - \tau^2/2 + \varepsilon(-\tau^2/2 + \tau^3/6) + \varepsilon^2(\tau^3/6 - \tau^4/24) + O(\varepsilon^3). \tag{18}$$

This expansion is uniformly valid in τ in the range $0 < \tau \ll 1/\varepsilon$, which is the range of physical interest since the projectile hits the ground well before $1/\varepsilon$ if $\varepsilon \ll 1$ (see below).

We wish to obtain the expansion in (18) without prior knowledge of the exact solution (17), using regular perturbation theory. We proceed similarly as for algebraic equations.

STEP A. Introduce the expansion

$$y(\tau) = y_0(\tau) + \varepsilon y_1(\tau) + \varepsilon^2 y_2(\tau) + O(\varepsilon^3), \tag{19}$$

where $y_0(\tau)$, $y_1(\tau)$, $y_2(\tau)$ are functions of τ to be determined.

STEP B. Substitute (19) into (16) (differential equation and initial conditions) written as

$$\frac{d^2y}{d\tau^2} + \varepsilon\frac{dy}{d\tau} + 1 = 0, \qquad y(0) = 0, \qquad \frac{dy}{d\tau}(0) - 1 = 0,$$

and expand the resulting equations in power series of ε. This gives

$$\begin{gathered}
\frac{d^2y_0}{d\tau^2} + 1 + \varepsilon\left(\frac{d^2y_1}{d\tau^2} + \frac{dy_0}{d\tau}\right) + \varepsilon^2\left(\frac{d^2y_2}{d\tau^2} + \frac{dy_1}{d\tau}\right) + O(\varepsilon^3) = 0,\\
y_0(0) + \varepsilon y_1(0) + \varepsilon^2 y_2(0) + O(\varepsilon^3) = 0,\\
\frac{dy_0}{d\tau}(0) - 1 + \varepsilon\frac{dy_1}{d\tau}(0) + \varepsilon^2\frac{dy_2}{d\tau}(0) + O(\varepsilon^3) = 0.
\end{gathered} \tag{20}$$

STEP C. Equate to zero the successive terms of the series in the left hand side of (6):

$$\begin{aligned}
&O(\varepsilon^0): && \frac{d^2y_0}{d\tau^2}+1=0, && y_0(0)=0, && \frac{dy_0}{d\tau}(0)-1=0,\\
&O(\varepsilon^1): && \frac{d^2y_1}{d\tau^2}+\frac{dy_0}{d\tau}=0, && y_1(0)=0, && \frac{dy_1}{d\tau}(0)=0,\\
&O(\varepsilon^2): && \frac{d^2y_2}{d\tau^2}+\frac{dy_1}{d\tau}, && y_2(0)=0, && \frac{dy_2}{d\tau}(0)=0,\\
&O(\varepsilon^3): && \cdots
\end{aligned} \tag{21}$$

STEP D. Successively solve the sequence of equations obtained in (21):

$$y_0(\tau)=\tau-\tau^2/2, \qquad y_1(\tau)=-\tau^2/2+\tau^3/6, \qquad y_2(\tau)=\tau^3/6-\tau^4/24. \tag{22}$$

Substituting (22) into (19) gives (18).

Exercise 2. Show that $y_n(\tau)=(-1)^n(\tau^{n+1}/(n+1)!-\tau^{n+2}/(n+2)!)$.

Exercise 3. The maximum altitude reached by the projectile is $y_\star := y(\tau_\star)$, where $\tau_\star$ is the time such that $dy(\tau_\star)/d\tau=0$ (why?). From (17) we have

$$\frac{dy}{d\tau}=\frac{(1+\varepsilon)e^{-\varepsilon\tau}}{\varepsilon}-\frac{1}{\varepsilon}.$$

Hence

$$\tau_\star=\frac{\ln(1+\varepsilon)}{\varepsilon}, \tag{23}$$

and

$$y_\star=y(\tau_\star)=\frac{1}{\varepsilon}-\frac{\ln(1+\varepsilon)}{\varepsilon^2}. \tag{24}$$

For small ε, $\tau_\star$ and $y_\star$ can be approximated by (recall that $\ln(1+z)=z-z^2/2+z^3/3+O(z^4)$),

$$\tau_\star=1-\varepsilon/2+\varepsilon^2/2+O(\varepsilon^3), \qquad y_\star=1/2-\varepsilon/3+\varepsilon^2/4+O(\varepsilon^3). \tag{25}$$

Re-obtain these expansions from (18) by perturbation theory.

Exercise 4. Check that (24) is consistent with the result we obtained for h using dimensional analysis.

Exercise 5. Recall that in the high friction limit, the appropriate equation for the projectile is

$$\frac{d^2z}{ds^2}=-\frac{dz}{ds}-\delta, \qquad z(0)=0, \quad \frac{dz}{ds}(0)=1, \tag{26}$$

where $\delta:=1/\varepsilon=mg/(kV)$. Here the altitude is measured in units of $\ell_2=Vm/k$, and the time in units of $\tau_2=m/k$. Solve this equation by regular perturbation technique when $0<\delta\ll 1$.

What is the maximal altitude reached by the projectile? What is the time of the flight? What is the ratio between the ascent and the descent times of the projectile?

Exercise 6. The dimensionless form of the equation for a perfect pendulum of length ℓ is

$$\frac{d^2\theta}{d\tau^2} = -\sin(\theta), \quad \theta(0) = \phi, \qquad \frac{d\theta}{d\tau}(0) = 0,$$

where the time is measured in units of $\sqrt{\ell/g}$. Solve this equation by regular perturbation when $0 < \phi \ll 1$. Is your approximation uniformly valid in time?